MACKEY FUNCTORS AND SHARPNESS FOR FUSION SYSTEMS

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ABSTRACT. We develop the fundamentals of Mackey functors in the setup of fusion systems including a parametrization and an explicit description of simple Mackey functors as well as some acyclicity conditions. For every exotic fusion system $\mathcal{F}$ on a finite p-group of p-rank two (p odd) we show that its homology decomposition with respect to $\mathcal{F}$-centric subgroups is sharp. This result extends Dwyer's sharpness results for finite groups.

1. INTRODUCTION.

In [3], Broto, Levi and Oliver introduced a family of topological spaces known as p-local finite groups, based on saturated fusion systems defined by Puig. These objects behave homotopically like the p-completed classifying space of a finite group but without necessarily being one of them. For instance, all these spaces have homology decompositions.

Let $\mathcal{F}$ be a saturated fusion system on the finite p-group S. By the groundbreaking recent work of Chermak [7], we can associate to each saturated fusion system $\mathcal{F}$ a unique p-local finite group that plays the role of the classifying space $B\mathcal{F}$ of $\mathcal{F}$. Among the homology decompositions of $B\mathcal{F}$ there is the subgroup decomposition for the family of $\mathcal{F}$-centric subgroups. It permits to reconstruct the classifying space of $\mathcal{F}$ by gluing together classifying spaces BP, where P runs over the collection of subgroups of S that are $\mathcal{F}$-centric.

For the finite group case, i.e., for $\mathcal{F} = \mathcal{F}_S(G)$ with G a finite group containing S as a Sylow p-subgroup, this homology decomposition possesses one further feature called *sharpness*. To explain this terminology we need first to consider the orbit category $\mathcal{O}(\mathcal{F})$ and the centric orbit category $\mathcal{O}(\mathcal{F}^c)$. Objects in $\mathcal{O}(\mathcal{F})$ are all subgroups of S and morphisms are given by $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Inn}(Q)\backslash \mathrm{Hom}_{\mathcal{F}}(P, Q)$, where $\mathrm{Inn}(Q)$ acts by post-composition. The centric orbit category $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ with objects the $\mathcal{F}$-centric subgroups of S. The aforementioned subgroup decomposition reads as

$$B\mathcal{F} \simeq \operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B},$$

where $\widetilde{B} \colon \mathcal{O}(\mathcal{F}^c) \to \mathsf{Top}$ is a functor such that $\widetilde{B}(P)$ has the homotopy type of BP for each $\mathcal{F}$-centric subgroup $P \leq S$. By the work of Bousfield and Kan [2], the homotopy colimit above gives rise to a first quadrant cohomology spectral sequence

$$E_2^{i,j} = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}^{i} H^j(-; \mathbb{F}_p) \Rightarrow H^{i+j}(B\mathcal{F}; \mathbb{F}_p).$$

Sharpness means that this spectral sequence collapses onto the vertical axis. This is equivalent to that the functor $H^j(-;\mathbb{F}_p)$ is *acyclic* for any $j \geq 0$, which means that the higher limits $\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)} H^j(-;\mathbb{F}_p)$ vanish for all $i \geq 1$ and $j \geq 0$. It was proven for the finite group case $\mathcal{F} = \mathcal{F}_S(G)$ by Dwyer in [12, Theorem 10.3], where it is referred to as *subgroup-sharpness for the collection of p-centric subgroups*. Note that there the result is stated over the p-subgroup orbit category $\mathcal{O}_p(G)$ and by [15, Lemma 2.1] the higher limits over $\mathcal{O}(\mathcal{F}^c)$ and $\mathcal{O}_p(G)$ coincide. This raises the following conjecture.

Conjecture A (Sharpness for fusion systems). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and consider the jth cohomology functor with trivial coefficients $\mathbb{F}_p$*

$$H^j(-;\mathbb{F}_p)\colon \mathcal{O}(\mathcal{F}^c) \to \mathbb{F}_p\text{-mod}$$

for any $j \geq 0$. Then the higher limits

$$\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)} H^j(-;\mathbb{F}_p)$$

vanish for all $i \geq 1$ and $j \geq 0$.

As stated before, this conjecture is known for finite groups, and it is strongly believed to hold for any saturated fusion system. In this paper we study the higher limits involved in the conjecture via Mackey functors. The connection is that Mackey functors on suitable categories have vanishing higher limits [14]. We obtain two main results, a general vanishing result and the positive confirmation of the conjecture for several exotic fusion systems.

For this task, we need to extend the definition of Mackey functors from finite groups to fusion systems. In Section 2, we introduce the appropriate notions of Mackey functor for both orbit categories $\mathcal{O}(\mathcal{F})$ and $\mathcal{O}(\mathcal{F}^c)$ and study their relation. Then we develop some standard properties of Mackey functors in Section 3, including a parametrization and a concrete description of simple Mackey functors. We discuss the acyclicity of Mackey functors in Section 4, and introduce the Mackeyfication functor in Section 5. This latter construction allows us to embed a given contravariant functor into a Mackey functor. This fact together with a change of coefficients argument gives the following theorem.

Theorem B (Theorem 5.2). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $N\colon \mathcal{O}(\mathcal{F}^c) \to \mathbb{F}_p\text{-mod}$ be a contravariant functor. If*

$$n = \max\{m \mid \exists P_0 < P_1 < \cdots < P_m \leq S, P_i \in \mathcal{F}^c\, \forall i\},$$

then

$$\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)} N = 0 \qquad \forall i \geq n+2.$$

In the above theorem, since $\mathcal{F}^c$ is closed under taking overgroups, the number n is determined by $\mathcal{F}$-centric subgroups Q of S of minimal order by $p^n = |S|/|Q|$. In the terminology of [3] we are claiming that the category $\mathcal{O}(\mathcal{F}^c)$ has *bounded limits over p*. Note that this was already known by [3, Corollary 3.3] but here we are able to give the explicit bound $n+2$.

The machinery developed in Section 3 provides us with a nice setting to study Conjecture A. Roughly, we proceed as follows: consider the jth cohomology functor $H^j(-;\mathbb{F}_p)$, or more generally a Mackey functor $M = (M^*, M_*)\colon \mathcal{O}(\mathcal{F}) \to \mathbb{F}_p\text{-mod}$ defined in the full orbit category (see Definition 2.4). Its restriction to the centric

orbit category $M|_{\mathcal{O}(\mathcal{F}^c)}$ might no longer be a Mackey functor, and so we cannot use the acyclicity of Mackey functors (Theorem 4.1) directly to conclude that $M^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic. Instead we break up the Mackey functor M by taking a composition series, then we study how the restrictions of the composition factors contribute to the higher limits of $M|_{\mathcal{O}(\mathcal{F}^c)}$. We obtain in Section 6 the following theorem.

Theorem C (Theorem 6.4). *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $M = (M^*, M_*) \colon \mathcal{O}(\mathcal{F}) \to \mathbb{F}_p\text{-}\mathsf{mod}$ be a Mackey functor. Then*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i M^*|_{\mathcal{O}(\mathcal{F}^c)} = 0 \qquad \forall i \geq 1$$

provided one of the following conditions hold.

(1) *$p = 2$ and $S \cong \mathrm{D}_{2^n}$, SD_{2^n}, or Q_{2^n}.*
(2) *p odd and $S \cong p_+^{1+2}$ is the extraspecial group of order p^3 and exponent p.*
(3) *$p = 3$ and $S \cong B(3, r; 0, \gamma, 0)$ is a maximal nilpotency class 3-group of 3-rank two.*
(4) *S is resistant.*

See Section 6 for a precise description of the group $B(3, r; 0, \gamma, 0)$. We consider the cases (2) and (3) because all saturated fusion systems on p-groups of p-rank two for p odd are classified by the first author and Ruiz and Viruel [11], and all exotic fusion systems among them are afforded by the p-groups in (2) and (3). The last case above (4) includes the abelian Sylow p-subgroup case and the proof does not rely on Mackey functors. Applying the cases (2) and (3) to the jth cohomology functor $M = H^j(-; \mathbb{F}_p)$ we have the following corollary.

Corollary D. *Let $\mathcal{F}$ be an exotic fusion system on a finite p-group S of p-rank two for p odd. Then $\mathcal{F}$ satisfies Conjecture A.*

Together with Dwyer's result for the group case, this shows that Conjecture A holds for any saturated fusion system $\mathcal{F}$ defined on a finite p-group of p-rank two with p odd. Dwyer's approach consists of applying the method of discarded orbits to a space acted by a finite group. In the context of exotic fusion systems this setup is no longer available and we have used instead Mackey functor technology. We believe this different strategy may be employed to study other exotic fusion systems as well. We plan to do this task in subsequent papers.

2. Mackey functors for fusion systems

In this section, we define Mackey functors for fusion systems as linear functors from a certain subquotient of the biset category defined by Bouc [5], or equivalently as certain restricted global Mackey functors defined by Webb [18].

Let k be a commutative ring with identity element. Denote by $\mathcal{I}$ the subcategory of the category of finite groups whose objects are the finite groups and morphisms are the group monomorphisms. Let $\mathcal{F}$ be a subcategory of $\mathcal{I}$ satisfying the following three conditions:

(1) $\mathrm{Ob}(\mathcal{F})$ is closed under taking subgroups.

(2) $\text{Mor}(\mathcal{F})$ is closed under taking restrictions and inverses.

(3) $\text{Mor}(\mathcal{F})$ contains all inner automorphisms of objects of $\mathcal{F}$.

For convenience, we call such a category $\mathcal{F}$ a *fusion system*. If, moreover, $\text{Ob}(\mathcal{F})$ consists of all subgroups of a fixed finite p-group S, then $\mathcal{F}$ is a *fusion system on S*, coinciding with the usual definition in the literature.

For finite groups P and Q, let $B_{\mathcal{F}}(Q, P)$ denote the subgroup of the double Burnside group $B(Q, P)$ generated by all isomorphism classes of (Q, P)-bisets of the form

$$[Q \times_\varphi P]$$

where φ is an $\mathcal{F}$-isomorphism from a subgroup, say U, of P to a subgroup of Q. Recall that the (Q, P)-biset $Q \times_\varphi P$ is obtained from the obvious (Q, P)-biset $Q \times P$ by identifying $(y\varphi(u), x) \sim (y, ux)$ for $x \in P, y \in Q, u \in U$.

Then we can define the category $\mathcal{D}_{\mathcal{F}}$ by setting $\text{Ob}(\mathcal{D}_{\mathcal{F}}) = \text{Ob}(\mathcal{F})$ and

$$\text{Hom}_{\mathcal{D}_{\mathcal{F}}}(P, Q) = kB_{\mathcal{F}}(Q, P) := k \otimes_{\mathbb{Z}} B_{\mathcal{F}}(Q, P)$$

for objects P, Q of $\mathcal{F}$ as a k-linear subcategory of the k-linearized biset category, thanks to the conditions on $\mathcal{F}$. We recall the composition law in $\mathcal{D} := \mathcal{D}_{\mathcal{I}}$. Note that $\mathcal{D}_{\mathcal{F}}$ is a k-linear subcategory of $\mathcal{D}$.

Lemma 2.1. *Let P, Q, R be finite groups. Let $U \leq P$, $V \leq Q$ and let $\varphi \colon U \to Q$, $\psi \colon V \to R$ be group monomorphisms. In the category $\mathcal{D}$, we have*

$$[R \times_\psi Q][Q \times_\varphi P] = \sum_{x \in [V \backslash Q / \varphi(U)]} [R \times_{\psi c_x \varphi} P]$$

where $[V \backslash Q / \varphi(U)]$ denotes a set of representatives of V-$\varphi(U)$-double cosets in Q and

$$\psi c_x \varphi \colon \varphi^{-1}(V^x \cap \varphi(U)) \xrightarrow{\varphi} V^x \cap \varphi(U) \xrightarrow{c_x} V \cap {}^x\varphi(U) \xrightarrow{\psi} \psi(V \cap {}^x\varphi(U)).$$

Now let $\mathcal{X}$ be a full subcategory of $\mathcal{F}$ *closed under $\mathcal{F}$-overconjugacy*, in the sense that if $\varphi \colon P \to Q$ is a morphism in $\mathcal{F}$ and P belongs to $\mathcal{X}$, then so does Q. For example, if $\mathcal{F}$ is a fusion system on a finite p-group S, then the full subcategory $\mathcal{F}^c$ of $\mathcal{F}$ consisting of the $\mathcal{F}$-centric subgroups of S is closed under $\mathcal{F}$-overconjugacy. Let $\mathcal{X}'$ denote the full subcategory of $\mathcal{F}$ with $\text{Ob}(\mathcal{X}') = \text{Ob}(\mathcal{F}) - \text{Ob}(\mathcal{X})$. Evidently, $\mathcal{X}'$ is *closed under $\mathcal{F}$-subconjugacy*, in the sense that if $\varphi \colon P \to Q$ is a morphism in $\mathcal{F}$ and Q belongs to $\mathcal{X}'$, then so does P. Consequently, $\mathcal{X}'$ is a fusion subsystem of $\mathcal{F}$ and the k-linear subcategory $\mathcal{D}_{\mathcal{F}, \mathcal{X}'}$ of $\mathcal{D}_{\mathcal{F}}$ with the same objects such that

$$\text{Hom}_{\mathcal{D}_{\mathcal{F}, \mathcal{X}'}}(P, Q) = kB_{\mathcal{X}'}(Q, P)$$

for all objects P, Q of $\mathcal{F}$ is an ideal of $\mathcal{D}_{\mathcal{F}}$. In fact,

Lemma 2.2. *Let $\mathcal{F}$ be a fusion system and $\mathcal{E}$ a subsystem of $\mathcal{F}$. Then $\mathcal{D}_{\mathcal{F}, \mathcal{E}}$ is an ideal of $\mathcal{D}_{\mathcal{F}}$ if and only if $\mathcal{E}$ is a full subcategory of $\mathcal{F}$ closed under $\mathcal{F}$-subconjugacy.*

Proof. Suppose that $\mathcal{D}_{\mathcal{F}, \mathcal{E}}$ is an ideal of $\mathcal{D}_{\mathcal{F}}$. Let $\varphi \colon P \to Q$ be a morphism in $\mathcal{F}$ and assume that Q is an object of $\mathcal{E}$. Let $R = \varphi(P)$ and let $\overline{\varphi} \colon P \to R$ denote the group isomorphism induced from φ. Then $\varphi = \iota_R^Q \circ \overline{\varphi}$ where $\iota_R^Q \colon R \to Q$ is the inclusion map and so $[Q \times_\varphi P] = [Q \times_{\iota_R^Q} R][R \times_{\overline{\varphi}} P]$. Since $\mathcal{E}$ is a fusion system and Q is an object of $\mathcal{E}$, we have that ι_R^Q is a morphism in $\mathcal{E}$. Since $\mathcal{D}_{\mathcal{F}, \mathcal{E}}$ is an ideal of $\mathcal{D}_{\mathcal{F}}$, it follows that $[Q \times_\varphi P] \in kB_{\mathcal{E}}(Q, P)$. Thus φ is a morphism in $\mathcal{E}$ and in particular P is an object of $\mathcal{E}$. This shows that $\mathcal{E}$ is a full subcategory of $\mathcal{F}$ closed under $\mathcal{F}$-subconjugacy. The converse is immediate from Lemma 2.1. $\qquad\square$

We denote by $\mathcal{D}^{\mathcal{X}} = \mathcal{D}_{\mathcal{F}}/\mathcal{D}_{\mathcal{F},\mathcal{X}'}$ the quotient category. Explicitly, $\mathrm{Ob}(\mathcal{D}^{\mathcal{X}}) = \mathrm{Ob}(\mathcal{F})$ and for objects P, Q of $\mathcal{F}$,

$$\mathrm{Hom}_{\mathcal{D}^{\mathcal{X}}}(P, Q) = kB_{\mathcal{F}}(Q, P)/kB_{\mathcal{X}'}(Q, P).$$

Since $\mathrm{Hom}_{\mathcal{D}^{\mathcal{X}}}(P, Q) = 0$ unless both P and Q belong to $\mathcal{X}$, we may and will assume that $\mathrm{Ob}(\mathcal{D}^{\mathcal{X}}) = \mathrm{Ob}(\mathcal{X})$. Note that $\mathrm{Hom}_{\mathcal{D}^{\mathcal{X}}}(P, Q)$ is a free k-module with basis elements the images of $[Q \times_{\varphi} P]$ where φ runs over representatives of (Q, P)-conjugacy classes of $\mathcal{F}$-isomorphisms from subgroups of P to subgroups of Q which are in $\mathcal{X}$.

We are now ready to give a definition of a Mackey functor for an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. In fact, we will give three equivalent definitions.

Definition 2.3. Let $\mathcal{F}$ be a fusion system and let $\mathcal{X}$ be an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. A *Mackey functor* for $\mathcal{X}$ over a commutative ring k with identity element is a k-linear (covariant) functor from $\mathcal{D}^{\mathcal{X}}$ to the category k-Mod of k-modules. Let

$$\mathsf{Mack}_k(\mathcal{X}) = \mathsf{Fun}_k(\mathcal{D}^{\mathcal{X}}, k\text{-Mod})$$

denote the category of Mackey functors for $\mathcal{X}$ over k.

The second definition of Mackey functor we present is a pair of functors (satisfying certain conditions) on the *orbit category* $\mathcal{O}(\mathcal{X})$ of $\mathcal{X}$, i.e., the category with the same objects as $\mathcal{X}$ and such that for $P, Q \in \mathrm{Ob}(\mathcal{X})$, the morphism set $\mathrm{Hom}_{\mathcal{O}(\mathcal{X})}(P, Q)$ is the set of $\mathrm{Inn}(Q)$-orbits of $\mathrm{Hom}_{\mathcal{X}}(P, Q)$. For a morphism $\varphi \colon P \to Q$ in $\mathcal{X}$, let $[\varphi]$ denote the $\mathrm{Inn}(Q)$-orbit of φ.

Definition 2.4. Let $\mathcal{F}$ be a fusion system and let $\mathcal{X}$ be an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. A *Mackey functor* for $\mathcal{X}$ over a commutative ring k with identity element is a pair of functors

$$M = (M^*, M_*) \colon \mathcal{O}(\mathcal{X}) \to k\text{-Mod}$$

satisfying the following conditions.

(1) (Bivariance) $M^* \colon \mathcal{O}(\mathcal{X}) \to k$-Mod is a contravariant functor, $M_* \colon \mathcal{O}(\mathcal{X}) \to k$-Mod is a covariant functor, and $M^*(P) = M_*(P) =: M(P)$ for all objects P of $\mathcal{X}$.

(2) (Isomorphism) $M^*([\alpha]) = M_*([\alpha]^{-1})$ if $[\alpha]$ is an isomorphism in $\mathcal{O}(\mathcal{X})$.

(3) (Mackey decomposition) If $Q, R \le P \le S$ are in $\mathcal{X}$, then

$$M^*([\iota_Q^P]) \circ M_*([\iota_R^P]) = \sum_{x \in [Q \backslash P / R]_{\mathcal{X}}} M_*([\iota_{Q \cap {}^x R}^Q]) \circ M^*([\iota_{Q \cap {}^x R}^{{}^x R}]) \circ M_*([c_x|_R]),$$

where $[Q \backslash P / R]_{\mathcal{X}} = \{x \in [Q \backslash P / R] \mid Q \cap {}^x R \in \mathcal{X}\}$.

Note that in the Mackey decomposition condition we take only those double coset representatives whose corresponding intersection belongs to $\mathcal{X}$. The equivalence of the two definitions of Mackey functors is well known and given as follows. Let $F \in \mathsf{Mack}_k(\mathcal{X})$. One can obtain a pair $M = (M^*, M_*)$ as in Definition 2.4 by setting $M(P) = F(P)$ for $P \in \mathrm{Ob}(\mathcal{X})$ and

$$M^*([\varphi]) = F([P \times_{\varphi^{-1}} Q]),$$
$$M_*([\varphi]) = F([Q \times_{\varphi} P]),$$

for $\varphi\colon P \to Q$ in $\mathcal{F}$. Conversely, given a pair $M = (M^*, M_*)$, as in Definition 2.4, a Mackey functor $F \in \mathsf{Mack}_k(\mathcal{X})$ is defined by setting $F(P) = M(P)$ for $P \in \mathrm{Ob}(\mathcal{X})$ and

$$F([Q \times_\varphi P]) = M_*([\varphi])M^*([\iota_U^P])$$

for $U \leq P$ in $\mathcal{X}$, $\varphi\colon U \to Q$ in $\mathcal{F}$.

Often it is convenient to use the following notations

$$\mathrm{r}_Q^P = M^*([\iota_Q^P])\colon M(P) \to M(Q),$$
$$\mathrm{t}_Q^P = M_*([\iota_Q^P])\colon M(Q) \to M(P),$$
$$\mathrm{iso}(\varphi) = M_*([\varphi]) = M^*([\varphi^{-1}])\colon M(P) \to M(\varphi(P)),$$

where $Q \leq P$ are in $\mathcal{X}$ and $\varphi\colon P \to \varphi(P)$ is an isomorphism in $\mathcal{X}$. It is easy to see that using these maps we can get yet another equivalent definition of Mackey functors which is close to the classical definition as follows.

Definition 2.5. Let $\mathcal{F}$ be a fusion system and let $\mathcal{X}$ be an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. A *Mackey functor* for $\mathcal{X}$ over a commutative ring k with identity element is a function

$$M\colon \mathrm{Ob}(\mathcal{X}) \to k\text{-}\mathsf{mod}$$

together with k-module homomorphisms

$$\mathrm{r}_Q^P\colon M(P) \to M(Q)$$
$$\mathrm{t}_Q^P\colon M(Q) \to M(P)$$
$$\mathrm{iso}(\varphi)\colon M(P) \to M(\varphi(P))$$

for all objects $Q \leq P$ in $\mathcal{X}$ and for all isomorphisms $\varphi\colon P \to \varphi(P)$ in $\mathcal{X}$ satisfying the following conditions.

(1) (Identity) $\mathrm{r}_P^P = \mathrm{t}_P^P = \mathrm{iso}(c_x|_P) = \mathrm{Id}_{M(P)}$ for P in $\mathcal{X}$ and for $x \in P$.
(2) (Transitivity) $\mathrm{r}_R^Q \circ \mathrm{r}_Q^P = \mathrm{r}_R^P$, $\mathrm{t}_Q^P \circ \mathrm{t}_R^Q = \mathrm{t}_R^P$, $\mathrm{iso}(\psi) \circ \mathrm{iso}(\varphi) = \mathrm{iso}(\psi \circ \varphi)$ for $R \leq Q \leq P$ in $\mathcal{X}$ and for isomorphisms φ, ψ in $\mathcal{X}$ such that $\psi \circ \varphi$ is defined.
(3) (Conjugation) $\mathrm{iso}(\varphi|_Q) \circ \mathrm{r}_Q^P = \mathrm{r}_{\varphi(Q)}^{\varphi(P)} \circ \mathrm{iso}(\varphi)$, $\mathrm{iso}(\varphi) \circ \mathrm{t}_Q^P = \mathrm{t}_{\varphi(Q)}^{\varphi(P)} \circ \mathrm{iso}(\varphi|_Q)$ for $Q \leq P$ in $\mathcal{X}$ and for isomorphisms $\varphi\colon P \to \varphi(P)$ in $\mathcal{X}$.
(4) (Mackey decomposition) For $Q, R \leq P$ in $\mathcal{X}$,

$$\mathrm{r}_Q^P \circ \mathrm{t}_R^P = \sum_{x \in [Q\backslash P/R]_{\mathcal{X}}} \mathrm{t}_{Q \cap {}^x R}^Q \circ \mathrm{r}_{Q \cap {}^x R}^{{}^x R} \circ \mathrm{iso}(c_x|_R).$$

Finally, we analyze the relation between Mackey functors for $\mathcal{F}$ and $\mathcal{X}$. From the canonical projection and inclusion $\mathcal{D}^{\mathcal{X}} = \mathcal{D}_{\mathcal{F}}/\mathcal{D}_{\mathcal{X}'} \leftarrow \mathcal{D}_{\mathcal{F}} \hookrightarrow \mathcal{D} = \mathcal{D}_{\mathcal{I}}$, we get functors

$$\mathsf{Mack}_k(\mathcal{X}) \hookrightarrow \mathsf{Mack}_k(\mathcal{F}) \xleftarrow{\mathrm{res}} \mathsf{Mack}_k := \mathsf{Mack}_k(\mathcal{I})$$

where the left embedding is induced by inflation and the right arrow by restriction. We denote by $\mathsf{Fun}^{\mathsf{bi}}(\mathcal{O}(\mathcal{X}), k\text{-}\mathsf{Mod})$ the category of those pairs (M^*, M_*) satisfying the Bivariance condition and the Isomorphism condition in Definition 2.4. Then

$\mathsf{Mack}_k(\mathcal{X})$ embeds into $\mathsf{Fun}^{\mathsf{bi}}(\mathcal{O}(\mathcal{X}), k\text{-}\mathsf{Mod})$ by forgetting the Mackey decomposition condition and we have the following commutative diagram

$$
\begin{array}{ccc}
\mathsf{Mack}_k(\mathcal{F}) & \longleftarrow & \mathsf{Mack}_k(\mathcal{X}) \\
\downarrow & & \downarrow \\
\mathsf{Fun}^{\mathsf{bi}}(\mathcal{O}(\mathcal{F}), k\text{-}\mathsf{Mod}) & \xrightarrow{\mathrm{res}} & \mathsf{Fun}^{\mathsf{bi}}(\mathcal{O}(\mathcal{X}), k\text{-}\mathsf{Mod}).
\end{array}
$$

For $M \in \mathsf{Mack}_k(\mathcal{F})$, its restriction $M|_{\mathcal{O}(\mathcal{X})}$ to $\mathcal{O}(\mathcal{X})$ is not necessarily a Mackey functor for $\mathcal{X}$; it is Mackey if and only if it satisfies the Mackey decomposition condition.

For example, the jth cohomology functor $H^j(-; k)$ with trivial coefficients k is a global Mackey functor, i.e., an object of Mack_k. So its restriction $H^j(-, k)|_{\mathcal{O}(\mathcal{F})}$ to $\mathcal{O}(\mathcal{F})$ is a Mackey functor for $\mathcal{F}$, but the further restriction $H^j(-, k)|_{\mathcal{O}(\mathcal{X})}$ to $\mathcal{O}(\mathcal{X})$ may not be a Mackey functor for $\mathcal{X}$ (see Example 4.4).

3. Simple Mackey functors.

In this section, we parametrize the simple Mackey functors for a fusion system and describe them explicitly. We mainly follow Bouc's approach and notations [5], but many of the results can be found in Webb's paper [18] too.

Let $\mathcal{F}$ be a fusion system and let k be a commutative ring with identity element. We begin by defining the k-linear functors

$$L_{Q,V} : \mathcal{D}_{\mathcal{F}} \to k\text{-}\mathsf{Mod},$$

that is, a Mackey functor for $\mathcal{F}$, where Q is an object of $\mathcal{F}$ and V is a simple $\mathrm{End}_{\mathcal{D}_{\mathcal{F}}}(Q)$-module. For each object P of $\mathcal{F}$, we set

$$L_{Q,V}(P) = \mathrm{Hom}_{\mathcal{D}_{\mathcal{F}}}(Q, P) \otimes_{\mathrm{End}_{\mathcal{D}_{\mathcal{F}}}(Q)} V,$$

and for each morphism $\rho\colon P \to P'$ in $\mathcal{D}_{\mathcal{F}}$, let

$$L_{Q,V}(\varphi)\colon L_{Q,V}(P) \to L_{Q,V}(P'), \quad \tau \otimes v \mapsto (\rho \circ \tau) \otimes v.$$

Then $L_{Q,V}$ is a Mackey functor for $\mathcal{F}$. Moreover

$$L_{Q,-}\colon \mathrm{End}_{\mathcal{D}_{\mathcal{F}}}(Q)\text{-}\mathsf{Mod} \to \mathsf{Mack}_k(\mathcal{F}), \quad V \mapsto L_{Q,V},$$

is a functor which is left adjoint to the evaluation functor

$$\mathrm{ev}_Q\colon \mathsf{Mack}_k(\mathcal{F}) \to \mathrm{End}_{\mathcal{D}_{\mathcal{F}}}(Q)\text{-}\mathsf{Mod}, \quad M \mapsto M(Q).$$

Using this adjunction, one can show that, for each simple $\mathrm{End}_{\mathcal{D}_{\mathcal{F}}}(Q)$-module V, the functor $L_{Q,V}$ has a unique maximal subfunctor $J_{Q,V}$ given by

$$J_{Q,V}(P) = \bigcap_{\rho \in \mathrm{Hom}_{\mathcal{D}_{\mathcal{F}}}(P,Q)} \ker(L_{Q,V}(\rho))$$

for each object P of $\mathcal{F}$. Let

$$S_{Q,V} = L_{Q,V}/J_{Q,V}$$

be the unique simple quotient of $L_{Q,V}$. We will write $S_{Q,V}^{\mathcal{F}} = S_{Q,V}$ when we want to indicate that this is a simple Mackey functor for $\mathcal{F}$.

For descriptions of simple Mackey functors, we introduce more notations following [6]. Let P, Q be objects of $\mathcal{F}$. Let $kI_{\mathcal{F}}(P, Q)$ denote the k-span of the basis

elements $[P \times_\varphi Q]$ of $kB_\mathcal{F}(P,Q)$ where $\varphi \colon U \to P$ is a morphism in $\mathcal{F}$ with $U < Q$, and set

$$k\overline{B}_\mathcal{F}(P,Q) = kB_\mathcal{F}(P,Q)/kI_\mathcal{F}(P,Q).$$

Note that $k\overline{B}_\mathcal{F}(P,Q)$ is a free k-module with basis consisting of the images of $[P \times_\alpha Q]$ where $\alpha \colon Q \to P$ are morphisms in $\mathcal{F}$, one for each element of $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(Q,P)$. In particular, $k\overline{B}_\mathcal{F}(Q,Q) \cong k\mathrm{Out}_\mathcal{F}(Q)$ as k-algebras. Thus $k\mathrm{Out}_\mathcal{F}(Q)$-modules can be viewed as modules for $kB_\mathcal{F}(Q,Q) = \mathrm{End}_{\mathcal{D}_\mathcal{F}}(Q)$ by inflation.

Now let Q be an object of $\mathcal{F}$ and let V be a simple $k\mathrm{Out}_\mathcal{F}(Q)$-module. We define the quotient functor

$$\overline{L}_{Q,V} \colon \mathcal{D}_\mathcal{F} \to k\text{-}\mathsf{Mod}$$

of $L_{Q,V}$ by setting $\overline{L}_{Q,V}(P) = k\overline{B}_\mathcal{F}(P,Q) \otimes_{k\,\mathrm{Out}_\mathcal{F}(Q)} V$. Note that $\overline{L}_{Q,V}(Q) \cong V \neq 0$. Since $J_{Q,V}$ is a unique maximal subfunctor of $L_{Q,V}$, its image $\overline{J}_{Q,V}$ in $\overline{L}_{Q,V}$ is also a unique maximal subfunctor of $\overline{L}_{Q,V}$. Note that $\bigcap_{\rho \in \mathrm{Hom}_{\mathcal{D}_\mathcal{F}}(-,Q)} \ker(\overline{L}_{Q,V}(\rho))$ is a proper subfunctor of $\overline{L}_{Q,V}$ that contains $\overline{J}_{Q,V}$. By the maximality of $\overline{J}_{Q,V}$ we deduce that they coincide and so

$$\overline{J}_{Q,V}(P) = \bigcap_{\rho \in \mathrm{Hom}_{\mathcal{D}_\mathcal{F}}(P,Q)} \ker(\overline{L}_{Q,V}(\rho))$$

for objects P of $\mathcal{F}$. Note that

$$S_{Q,V} = L_{Q,V}/J_{Q,V} \cong \overline{L}_{Q,V}/\overline{J}_{Q,V}.$$

and that $S_{Q,V}(Q) \cong V$ but $S_{Q,V}(R) = 0$ for all $R < Q$. In this sense, Q is a *minimal group* for the Mackey functor $S_{Q,V}$.

Proposition 3.1. *Let $\mathcal{F}$ be a fusion system and let k be a commutative ring with identity element. Let $\mathcal{X}$ be an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. The functor $S_{Q,V}^\mathcal{F}$ where Q is an object of $\mathcal{F}$ and V is a simple $k\mathrm{Out}_\mathcal{F}(Q)$-module belongs to $\mathsf{Mack}_k(\mathcal{X})$ if and only if $Q \in \mathcal{X}$. Moreover, every simple Mackey functor in $\mathsf{Mack}_k(\mathcal{X})$ is of this form. If R is an object of $\mathcal{X}$ and W is a simple $k\mathrm{Out}_\mathcal{F}(R)$-module, then $S_{Q,V}^\mathcal{F} \cong S_{R,W}^\mathcal{F}$ if and only if there is an $\mathcal{F}$-isomorphism $\alpha \colon Q \to R$ such that $^\alpha V \cong W$, where $^\alpha V$ is the $k\mathrm{Out}_\mathcal{F}(R)$-module such that $^\alpha V = V$ as a k-module and for $\varphi \in \mathrm{Aut}_\mathcal{F}(R)$, $[\varphi] \cdot v = [\alpha^{-1}\varphi\alpha] \cdot v$ where $\varphi \in \mathrm{Aut}_\mathcal{F}(R)$ and $v \in V$.*

Proof. We first prove that that $S_{Q,V}^\mathcal{F} \in \mathsf{Mack}_k(\mathcal{X})$ if and only if $Q \in \mathrm{Ob}(\mathcal{X})$. If Q is an object of $\mathcal{X}$, then $\mathrm{Hom}_{\mathcal{D}_{\mathcal{F},\mathcal{X}'}}(P,P') = kB_{\mathcal{X}'}(P',P)$ acts trivially on $\overline{L}_{Q,V}^\mathcal{F}(P) = k\overline{B}_\mathcal{F}(P,Q) \otimes_{k\,\mathrm{Out}_\mathcal{F}(Q)} V$ because $kB_{\mathcal{X}'}(P',P)k\overline{B}_\mathcal{F}(P,Q) \subseteq k\overline{B}_{\mathcal{X}'}(P',Q) = 0$. Thus $\mathcal{D}_{\mathcal{F},\mathcal{X}'}$ acts trivially on $\overline{L}_{Q,V}^\mathcal{F}$ and hence on $S_{Q,V}^\mathcal{F}$. This shows that $S_{Q,V}^\mathcal{F} \in \mathsf{Mack}_k(\mathcal{X})$. Conversely, suppose that $S_{Q,V}^\mathcal{F} \in \mathsf{Mack}_k(\mathcal{X})$. Then $kB_{\mathcal{X}'}(Q,Q)$ acts trivially on $S_{Q,V}^\mathcal{F}(Q) \cong V$. Thus $kB_{\mathcal{X}'}(Q,Q) \subsetneq kB_\mathcal{F}(Q,Q)$, so $Q \in \mathrm{Ob}(\mathcal{X})$.

To classify the simple functors of $\mathsf{Mack}_k(\mathcal{F})$ the proof in [5, Theorem 4.3.10] can be applied. For the general case, it is enough to note that the inflation functor $\mathsf{Mack}_k(\mathcal{X}) \hookrightarrow \mathsf{Mack}_k(\mathcal{F})$ preserves simple functors. $\qquad\square$

The following result is presented in [18, Theorem 2.6] for global Mackey functors and in [5, Theorem 4.3.20] for general biset functors.

Proposition 3.2. *Let $\mathcal{F}$ be a fusion system and let k be a commutative ring with identity element. Let Q be an object of $\mathcal{F}$ and let V be a simple $k\operatorname{Out}_{\mathcal{F}}(Q)$-module. The value of the simple Mackey functor $S_{Q,V}^{\mathcal{F}}$ for $\mathcal{F}$ over k at an object P of $\mathcal{F}$ is given by*

$$S_{Q,V}^{\mathcal{F}}(P) \cong \bigoplus_{Q \overset{\alpha}{\cong} L \leq_P P} \operatorname{tr}_L^{N_P(L)}({}^{\alpha}V).$$

For objects $R \leq P$ of $\mathcal{F}$, we have

$$
S_{Q,V}^{\mathcal{F}}(P) \xrightarrow{\mathrm{r}_R^P} S_{Q,V}^{\mathcal{F}}(R) \qquad S_{Q,V}^{\mathcal{F}}(R) \xrightarrow{\mathrm{t}_R^P} S_{Q,V}^{\mathcal{F}}(P)
$$
$$
v \in \mathrm{t}_L^{N_P(L)}({}^{\alpha}V) \;\mapsto\; (w_\beta)_\beta, \qquad w \in \mathrm{t}_{L'}^{N_R(L')}({}^{\beta}V) \;\mapsto\; (v_\alpha)_\alpha,
$$

where $\alpha\colon Q \cong L \leq_P P$, $\beta\colon Q \cong L' \leq_R R$ and

$$
w_\beta = \begin{cases} [\beta^{-1}c_x\alpha]v, & \text{if } {}^xL = \beta(Q) \text{ for some } x \in P, \\ 0, & \text{otherwise}; \end{cases}
$$
$$
v_\alpha = \begin{cases} \mathrm{t}_{N^x_R(\alpha(Q))}^{N_P(\alpha(Q))}([\alpha^{-1}c_x\beta]w), & \text{if } {}^xL' = \alpha(Q) \text{ for some } x \in P, \\ 0, & \text{otherwise.} \end{cases}
$$

In this proposition, the direct sum is taken over the P-conjugacy classes of the subgroups of P that are $\mathcal{F}$-conjugate to Q. For each of these classes a representative L is chosen and an $\mathcal{F}$-isomorphism $\alpha\colon Q \to L$ is also fixed. The map $\operatorname{tr}_L^{N_P(L)}\colon {}^{\alpha}V \to {}^{\alpha}V$ is the relative trace map, where $N_P(L)$ acts on the $k\operatorname{Out}_{\mathcal{F}}(L)$-module ${}^{\alpha}V$ via the map $N_P(L) \to \operatorname{Out}_{\mathcal{F}}(L)$ given by conjugation.

It is a well known fact that every simple biset functor over a commutative ring k takes as its values finite dimensional vector spaces over a residue field of k. Bouc gave a proof of this fact in [5, Remark 4.4.4] combining a more general version [5, Theorem 4.3.20] of Proposition 3.2 with an argument [5, Lemma 4.4.2, Corollary 4.4.3] in commutative algebra. Here we give a shorter proof for that argument.

Lemma 3.3. *Let R be a commutative ring with identity element and let A be an R-algebra which is finitely generated as an R-module. Every simple A-module M is a finite dimensional vector space over a field R/I for some maximal ideal I of R.*

Proof. Since A is finitely generated as an R-module and V is a simple A-module and hence generated by any nonzero element, V is finitely generated as an R-module. So it suffices to show that there is a maximal ideal I of R such that $IV = 0$. For each maximal ideal I of R, IV is an A-submodule of V so that either $IV = 0$ or $V = IV$. Suppose that $V = IV$ for all maximal ideals I of R. Localize $V = IV$ at I and apply Nakayama's lemma to get that the localization of V at every maximal ideal I of R is 0, whence $V = 0$, a contradiction. Thus $IV = 0$ for some maximal ideal I of R, as desired. $\square$

Thévenaz and Webb [17, Proposition 16.10] showed that a simple Mackey functor for a fixed finite group G over a field of characteristic p is cohomological if and only if its minimal subgroups are p-groups. We show that the same is true for simple Mackey functors for fusion systems. First we recall a group theoretic result.

Lemma 3.4. *Let G be a finite group, H a subgroup of G and Q a p-subgroup of G. Then*

$$|N_G(Q,H) : H| \equiv |G : H| \mod p,$$

where $N_G(Q,H) = \{x \in G \mid {}^xQ \leq H\}$.

Proof. Consider the right Q-action on the set $H\backslash G$ of right H-cosets in G. The Q-fixed points are $H\backslash N_G(Q,H)$ and the nontrivial Q-orbits have cardinality divisible by p. □

Proposition 3.5. *Let $\mathcal{F}$ a fusion system and let k be a field of characteristic p. Let Q be a finite group and V a simple $k\operatorname{Out}_{\mathcal{F}}(Q)$-module. The simple Mackey functor $S_{Q,V}$ for $\mathcal{F}$ over k is cohomological if and only if Q is a p-group.*

Proof. This proposition can be obtained as a corollary from Proposition 3.2. Here we give a proof which does not use this proposition.

First suppose that Q is a p-group. We show that $\overline{L}_{Q,V}$ is cohomological. Then $S_{Q,V}$ is also cohomological, because it is a quotient of $\overline{L}_{Q,V}$. Let $R \leq P$ be objects in $\mathcal{F}$. It suffices to show that

$$[P \times_R P][P \times_\varphi Q] \equiv |P : R|[P \times_\varphi Q] \mod kI_{\mathcal{F}}(P,Q)$$

for every $\mathcal{F}$-morphism $\varphi\colon Q \to P$, where $[P \times_R P] := [P \times_{\iota_R^P} P]$. Since $[P \times_\varphi Q] = [P \times_{\varphi(Q)} \varphi(Q)][\varphi(Q) \times_\varphi Q]$, we only need to prove the above equation when $Q \leq P$ and $\varphi\colon Q \to P$ is the inclusion, i.e., it suffices to show that

$$[P \times_R P][P \times_Q Q] \equiv |P : R|[P \times_Q Q] \mod kI_{\mathcal{F}}(P,Q)$$

for $Q \leq P$. We have

$$\begin{aligned}
(\text{LHS}) &= \sum_{x \in [R\backslash P/Q]} [P \times_{R^x \cap Q} Q] \\
&\equiv \sum_{x \in [R\backslash N_P(Q,R)/Q]} [P \times_Q Q] \\
&= |N_P(Q,R) : R|[P \times_Q Q] \\
&= |P : R|[P \times_Q Q] = (\text{RHS}).
\end{aligned}$$

The first equality is a usual double coset decomposition. The second equivalence comes from that we have $R^x \cap Q = Q$ if and only if $Q \leq R^x$. For the third equality, note that $R\backslash N_P(Q,R)/Q = R\backslash N_P(Q,R)$. Finally the last equality follows from Lemma 3.4.

Conversely, suppose that $S := S_{Q,V}$ is cohomological. Let R be a Sylow p-subgroup of Q. Then

$$S(Q) \xrightarrow{\mathrm{r}_R^Q} S(R) \xrightarrow{\mathrm{t}_R^Q} S(Q)$$

is multiplication by $|Q : R|$ and hence an isomorphism. Since $S(Q) \cong V \neq 0$, we have $S(R) \neq 0$. Since Q is a minimal group of S, it follows that $Q = R$, so Q is a p-group. □

We record some immediate consequences of the previous propositions for later use. The first two parts of the following corollary appear in [18, Proposition 2.8, Corollary 2.9].

Corollary 3.6. *Let $\mathcal{F}$ a fusion system whose objects are finite p-groups. Suppose that k is a field of characteristic p. Let Q, P be objects of $\mathcal{F}$ and let V be a simple $k\operatorname{Out}_{\mathcal{F}}(Q)$-module. Consider the simple Mackey functor $S_{Q,V}$ for $\mathcal{F}$ over k.*

(1) If $S_{Q,V}(P) \neq 0$, then there exists a morphism $\alpha\colon Q \to P$ in $\mathcal{F}$ such that the stabilizer $\operatorname{Stab}_P({}^\alpha V) = \alpha(Q)$; in particular, $C_P(\alpha(Q)) \leq \alpha(Q)$.

(2) We have $S_{Q,k}(P) \neq 0$ if and only if Q is $\mathcal{F}$-conjugate to P, where k is the trivial $k \operatorname{Out}_{\mathcal{F}}(Q)$-module.

(3) Suppose that $O_p(\operatorname{Out}(Q)) \in \operatorname{Syl}_p(\operatorname{Out}(Q))$. Then we have $S_{Q,V}(P) \neq 0$ if and only if Q is $\mathcal{F}$-conjugate to P.

(4) If $R < P$, then $\operatorname{t}_R^P \operatorname{r}_R^P = 0$ on $S_{Q,V}(P)$.

Proof. (1) By Proposition 3.2, if $S_{Q,V}(P) \neq 0$ there must be some $\mathcal{F}$-isomorphism $\alpha \colon Q \to L$ with $L \leq P$ such that $\operatorname{tr}_L^{N_P(L)}(^{\alpha}V) \neq 0$. Because of the transitivity of the relative trace, we must also have that $\operatorname{tr}_L^{\operatorname{Stab}_P(^{\alpha}V)}(^{\alpha}V) \neq 0$. As $\operatorname{Stab}_P(^{\alpha}V) \leq P$ is a p-group this is only possible if $\operatorname{Stab}_P(^{\alpha}V) = L$. Moreover, as the action of $C_P(L)$ on $^{\alpha}V$ is trivial we deduce that $C_P(L) \leq L$.

(2) Clearly if Q is $\mathcal{F}$-conjugate to P then $S_{Q,k}(P) \cong S_{Q,k}(Q) \cong k \neq 0$. Conversely, suppose that $S_{Q,k}(P) \neq 0$. By (1), there is some $\mathcal{F}$-isomorphism $\alpha \colon Q \to L$ with $L \leq P$ such that $\operatorname{tr}_L^{N_P(L)}(^{\alpha}V) \neq 0$. Note that $\operatorname{Stab}_P(^{\alpha}k) = N_P(L) = L$ as k is the trivial module and hence that, as P is a p-group, $P = L$.

(3) We have $S_{Q,V}(Q) = V \neq 0$ and so $S_{Q,V}(P) \neq 0$ if Q is $\mathcal{F}$-conjugate to P. Conversely, suppose that $S_{Q,V}(P) \neq 0$. By (1), it follows that Q is $\mathcal{F}$-conjugate to a subgroup of P. Now suppose that $|Q| < |P|$. Then for each $\mathcal{F}$-isomorphism $\alpha \colon Q \to L$ with $L \leq P$, we have in fact $L < P$ and so $L < N_P(L)$. Since $O_p(\operatorname{Out}(Q)) \in \operatorname{Syl}_p(\operatorname{Out}(Q))$, we have $O_p(\operatorname{Out}(L)) \in \operatorname{Syl}_p(\operatorname{Out}(L))$ and hence $O_p(\operatorname{Out}_{\mathcal{F}}(L)) \in \operatorname{Syl}_p(\operatorname{Out}_{\mathcal{F}}(L))$. Then since $^{\alpha}V$ is a simple $k\operatorname{Out}_{\mathcal{F}}(L)$-module, $O_p(\operatorname{Out}_{\mathcal{F}}(L))$ acts trivially on $^{\alpha}V$ by Clifford's theorem. In particular, $N_P(L)$ acts trivially on $^{\alpha}V$. Thus $\operatorname{tr}_L^{N_P(L)}(^{\alpha}V) = 0$. By Proposition 3.2, it follows that $S_{Q,V}(P) = 0$, a contradiction. So $|Q| = |P|$ and hence Q is $\mathcal{F}$-conjugate to P, as desired.

(4) This is a consequence of Proposition 3.5. $\qquad\square$

4. ACYCLICITY OF MACKEY FUNCTORS

In this section, we present an important acyclicity condition for Mackey functors for fusion systems.

Theorem 4.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then for every Mackey functor $M \colon \mathcal{O}(\mathcal{F}^c) \to \mathbb{Z}_{(p)}\text{-}\mathsf{Mod}$, the contravariant part M^* is acyclic.*

This theorem is a consequence of the combination of works of Jackowski and McClure [14] and Díaz and Libman [8]. In [14], Jackowski and McClure, following Dress's definition of Mackey functors for a finite group G, define a *Mackey functor* on a small category $\mathcal{C}$ with pullbacks as a bivariant functor $M \colon \mathcal{C} \to \mathbb{Z}\text{-}\mathsf{Mod}$ such that every pullback diagram

$$\begin{array}{ccc} W & \xrightarrow{\gamma} & Y \\ \delta \downarrow & & \downarrow \beta \\ X & \xrightarrow{\alpha} & Z \end{array}$$

in $\mathcal{C}$ induces a commutative diagram

$$
\begin{array}{ccc}
M(W) & \xrightarrow{M_*(\gamma)} & M(Y) \\
{\scriptstyle M^*(\delta)}\uparrow & & \uparrow{\scriptstyle M^*(\beta)} \\
M(X) & \xrightarrow{M_*(\alpha)} & M(Z).
\end{array}
$$

Further they define a *proto-Mackey functor* on a small category $\mathcal{B}$ as a a bivariant functor $M : \mathcal{B} \to \mathbb{Z}\text{-Mod}$ such that its additive extension $M_{\mathrm{II}} : \mathcal{B}_{\mathrm{II}} \to \mathbb{Z}\text{-Mod}$ is a Mackey functor in the previous sense, provided that $\mathcal{B}_{\mathrm{II}}$ has pullbacks. Then they show the following acyclicity criterion.

Proposition 4.2 ([14, Corollary 5.16]). *Let $\mathcal{B}$ be a small category satisfying the following conditions.*

- (B0) *The product of each pair of objects of $\mathcal{B}$ and the pullback of each diagram $c \to e \leftarrow d$ of objects of $\mathcal{B}$ exist in $\mathcal{B}_{\mathrm{II}}$.*
- (B1) *$\mathcal{B}$ has finitely many isomorphism classes of objects, each set of morphisms is finite, and all self-maps in $\mathcal{B}$ are isomorphisms.*
- (B2) *For each object P of $\mathcal{B}$ there is an object Q of $\mathcal{B}$ with $|\operatorname{Hom}_{\mathcal{B}}(P,Q)|$ prime to p.*

Then every proto-Mackey functor $M : \mathcal{B} \to \mathbb{Z}_{(p)}\text{-Mod}$ is acyclic.

The condition (B0), which is denote by $(\mathrm{PB}\times_{\mathrm{II}})$ in [14], ensures that $\mathcal{B}_{\mathrm{II}}$ has pullbacks and products of pairs of objects which are compatible with coproducts. This allows them to define Mackey functors and the Burnside ring for $\mathcal{B}_{\mathrm{II}}$. Using the action of the Burnside ring on Mackey functors and the conditions (B1) and (B2), they prove the above proposition.

On the other hand, Díaz and Libman [8] construct the Burnside ring of $\mathcal{O}(\mathcal{F}^c)_{\mathrm{II}}$ using the products in $\mathcal{O}(\mathcal{F}^c)_{\mathrm{II}}$, which goes back to Puig. It is easy to show that $\mathcal{O}(\mathcal{F}^c)_{\mathrm{II}}$ has equalizers. Consequently, $\mathcal{O}(\mathcal{F}^c)_{\mathrm{II}}$ has pullbacks.

Proposition 4.3 ([8, Propositions 2.9 and 2.10]). *Suppose that $\mathcal{F}$ is a saturated fusion system on a finite p-group S. Then $\mathcal{O}(\mathcal{F}^c)$ satisfies the condition* (B0). *In particular, for $\mathcal{F}$-centric subgroups $Q, R \le P \le S$, the pullback of $R \xrightarrow{[\iota_R^P]} P \xleftarrow{[\iota_Q^P]} Q$ is given by*

$$
\begin{array}{ccc}
\coprod_x Q \cap {}^x R & \xrightarrow{([\iota_{Q \cap {}^x R}^Q])} & Q \\
{\scriptstyle ([c_{x^{-1}}])}\downarrow & & \downarrow{\scriptstyle [\iota_Q^P]} \\
R & \xrightarrow{[\iota_R^P]} & P
\end{array}
$$

where the coproduct is taken over $x \in [Q \backslash P / R]_{\mathcal{F}^c}$.

In fact, $\mathcal{O}(\mathcal{F}^c)$ also satisfies the remaining two conditions. The condition (B1) is trivially satisfied and the condition (B2) is a consequence of the well known fact that $|\operatorname{Hom}_{\mathcal{O}(\mathcal{F}^c)}(P, S)|$ is prime to p for every $\mathcal{F}$-centric $P \le S$. Consequently, every proto-Mackey functor $M : \mathcal{O}(\mathcal{F}^c) \to \mathbb{Z}_{(p)}\text{-Mod}$ is acyclic. By [14, Lemma 5.13], proto-Mackey functors on $\mathcal{O}(\mathcal{F}^c)$ are in fact the same as Mackey functors for $\mathcal{F}^c$ as defined in this paper. This proves Theorem 4.1.

The following example shows that although being Mackey is a sufficient condition for acyclicity it is not a necessary condition.

Example 4.4 (Díaz–Libman). We will exhibit a finite p-group S such that the degre 1 cohomology functor $H^1(-;\mathbb{F}_p)\colon \mathcal{O}(\mathcal{F}^c) \to \mathbb{F}_p\text{-mod}$ is not a Mackey functor for the nilpotent fusion system $\mathcal{F} = \mathcal{F}_S(S)$. By the sharpness results of Dwyer [12] we know that $\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)} H^1(-;\mathbb{F}_p) = 0$ for $i \geq 1$ though.

The p-group S will have order p^{p+3} (p any prime) and two centric subgroups P, $Q \leq S$ such that:

(a) $PQ = S$, $P \cap Q$ is not centric.

(b) $H^1(Q;\mathbb{F}_p) \xrightarrow{\text{t}} H^1(S;\mathbb{F}_p) \xrightarrow{\text{r}} H^1(P;\mathbb{F}_p)$ is nonzero.

This is enough to prove that the subgroup decomposition for $\mathcal{F}_S(S)$ is not Mackey.

Consider first the Jordan block matrix A_n of size n and eigenvalue 1. It is easy to check that, for a prime p, A_p has order p working in $\mathbb{F}_p$. Now define the $(p+2) \times (p+2)$ matrix with two Jordan blocks, one of size p and one of size 2:

$$B_p = \begin{pmatrix} A_p & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then B_p also has order p working in $\mathbb{F}_p$, and the eigenspace associated to eigenvalue 1 consists of all vectors $\begin{pmatrix} * & 0 & \ldots & 0 & * & 0 \end{pmatrix}^t$, where ^{t} stands for transposition.

Define $Q = C_p \times C_p \times \cdots \times C_p$ ($(p+2)$ copies) and $S = Q \rtimes \langle B_p \rangle$. Notice that the subspace U of Q with last coordinate equal to zero is invariant under B_p. Define $P = U \rtimes \langle B_p \rangle$. It is clear than $PQ = S$ and that $P \cap Q = U$ is not centric as $Q \leq C_S(U)$. Subgroup Q is centric, and subgroup P is centric because $C_S(P) \leq C_S(\langle B_p \rangle) \leq P$. So we are left to prove that transfer from Q followed by restriction to P is nonzero.

Recall that the transfer $\text{t} : H^1(H;\mathbb{F}_P) \to H^1(G;\mathbb{F}_p)$, where $H \leq G$, is given by

$$\text{t}(f)(x) = \sum_i f(t_{\sigma_i}^{-1} x t_i),$$

where $\{t_i\}$ are representatives for the cosets G/H and σ is the permutation of the representatives induced by multiplication by x on the cosets G/H.

Our situation is $H^1(Q;\mathbb{F}_p) \xrightarrow{\text{t}} H^1(Q \rtimes \langle B_p \rangle;\mathbb{F}_p) \xrightarrow{\text{r}} H^1(U \rtimes \langle B_p \rangle;\mathbb{F}_p)$. Consider an arbitrary homomorphism $f : Q = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \to \mathbb{F}_p$ and the element $u \in U \leq U \rtimes \langle B_p \rangle$ given by $u = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & 0 \end{pmatrix}^t$. Then $\text{r}(\text{t}(f))(u) = \sum_{i=0}^{p-1} f(t_i^{-1} u t_i)$ because the permutation induced by u on the cosets is the identity (as Q is normal in S and $u \in Q$). Taking $I, B_p, \cdots, B_p^{p-1}$ as representatives we obtain

$$\text{r}(\text{t}(f))(u) = \sum_{i=0}^{p-1} f((B_p)^i(u)) = \sum_{i=0}^{p-1} f((A_p)^i(v)) = f((I + A_p + \cdots + A_p^{p-1})v),$$

where $v = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^t$. Hence we have to consider the sum $C = I + A_p + \cdots + A_p^{p-1}$. Write $c = \begin{pmatrix} c_1 & c_2 & \ldots & c_{p-1} & c_p \end{pmatrix}^t$ for the rightmost column of C. Then a direct computation shows that $c_2 = \cdots = c_p = 0$ in $\mathbb{F}_p$ and $c_1 = p - 1$ (this last equality hold integrally). Hence we deduce that

$$\text{r}(\text{t}(f))(u) = f(Cv) = f(C \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^t) = f(\begin{pmatrix} p-1 & 0 & \ldots & 0 & 0 \end{pmatrix}^t).$$

Choosing $f : \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p \to \mathbb{F}_p$ which does not vanish on the first coordinate we are done.

Now we present a strategy to study acyclicity of restrictions of Mackey functors.

Proposition 4.5. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let k be a field of characteristic p. Consider a Mackey functor $M \colon \mathcal{O}(\mathcal{F}) \to k\text{-}\mathsf{mod}$ which takes finite dimensional k-vector spaces as its values. If for each composition factor $S_{Q,V}$, where $Q \leq S$ is non-$\mathcal{F}$-centric and V is a simple $k \operatorname{Out}_{\mathcal{F}}(Q)$-module, the functor $S_{Q,V}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic, then $M^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic.*

Proof. By assumption there is a finite-length filtration of M by successive maximal Mackey subfunctors

$$0 = M_r \subset M_{r-1} \subset \ldots \subset M_1 \subset M_0 = M.$$

Restricting this filtration we get a filtration of bivariant functors $N_i = M_i|_{\mathcal{O}(\mathcal{F}^c)}$. From the long exact sequence of higher limits associated to the short exact sequences $0 \to N_{i+1}^* \to N_i^* \to N_i^*/N_{i+1}^* \to 0$, it is immediate that if each N_i^*/N_{i+1}^* is acyclic then so is $N_0^* = M^*|_{\mathcal{O}(\mathcal{F}^c)}$. Now, by Proposition 3.1, each quotient M_i/M_{i+1} is of the form $S_{Q,V}$, where $Q \leq S$ is a subgroup of S and V is a simple $k \operatorname{Out}_{\mathcal{F}}(Q)$-module. Hence, each quotient N_i^*/N_{i-1}^* is of the form $S_{Q,V}^*|_{\mathcal{O}(\mathcal{F}^c)}$. If Q is $\mathcal{F}$-centric then $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)}$ is in $\mathsf{Mack}_k(\mathcal{F}^c)$ by Proposition 3.1 and so $S_{Q,V}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic by Proposition 4.1. Therefore, $M^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic provided that $S_{Q,V}^*|_{\mathcal{O}(\mathcal{F}^c)}$ is acyclic whenever Q is not $\mathcal{F}$-centric. $\square$

The next proposition together with Proposition 4.1 gives sufficient conditions for acyclicity, which may be applied to functors $S_{Q,V}^*|_{\mathcal{O}(\mathcal{F}^c)}$ with Q non-$\mathcal{F}$-centric as in the statement of Proposition 4.5. In its statement and proof we use the notations r, t and iso introduced in Section 2.

Proposition 4.6. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let k be a field of characteristic p. Consider a Mackey functor M for $\mathcal{F}$ over k. Then the restriction $M|_{\mathcal{O}(\mathcal{F}^c)}$ is a Mackey functor for $\mathcal{F}^c$ if the composite*

$$M(P) \xrightarrow{\mathrm{r}_{P \cap R}^P} M(P \cap R) \xrightarrow{\mathrm{t}_{P \cap R}^R} M(R)$$

is zero whenever $P, R \leq S$ are $\mathcal{F}$-centric and $P \cap R$ is not $\mathcal{F}$-centric.

Proof. Since M is a Mackey functor for $\mathcal{F}$, we have

$$\mathrm{r}_R^T \, \mathrm{t}_P^T = \sum_{x \in [R \backslash T / P]} \mathrm{t}_{R \cap {}^x P}^R \, \mathrm{r}_{R \cap {}^x P}^{{}^x P} \, \mathrm{iso}(c_x|_P)$$

for all $P, R \leq T \leq S$. As remarked at the end of Section 2, the restriction $M|_{\mathcal{O}(\mathcal{F}^c)}$ is a Mackey functor for $\mathcal{F}^c$ if the above equality holds with the sum replaced by the sum over $x \in [P \backslash T / R]_{\mathcal{F}^c}$ whenever P, R, T are $\mathcal{F}$-centric. In particular, this is the case provided that

$$\mathrm{t}_{R \cap {}^x P}^R \, \mathrm{r}_{R \cap {}^x P}^{{}^x P} \, \mathrm{iso}(c_x|_P) = 0$$

whenever $P \cap {}^x R$ is not $\mathcal{F}$-centric. The proposition follows. $\square$

5. Mackeyfication and bound on limit

In [4, Lemma 2.10], Bouc constructed a left adjoint functor to the forgetful functor sending a Mackey functor for a fixed finite group G to its contravariant part. We show that this construction carries over to our situation and as a consequence get a vanishing of higher limits result.

Theorem 5.1. *Let $\mathcal{F}$ be a fusion system and let $\mathcal{X}$ be an $\mathcal{F}$-overconjugacy closed full subcategory of $\mathcal{F}$. Let k be a commutative ring with identity element. The forgetful functor to the contravariant part*

$$U^*\colon \mathsf{Mack}_k(\mathcal{X}) \to \mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod}), \quad M \mapsto M^*,$$

has a left adjoint

$$I\colon \mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod}) \to \mathsf{Mack}_k(\mathcal{X}).$$

Proof. Let N be an object of $\mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod})$. We define an object $I(N)$ of $\mathsf{Mack}_k(\mathcal{X})$ as follows, using Definition 2.4. The idea is to formally adjoin 'terms from below' to N, which will play the role of the image of transfer. For each object P of $\mathcal{X}$, consider the direct sum $\bigoplus_L N(L)$ where L runs over all $L \leq P$ belonging to $\mathcal{X}$. The group P acts on this direct sum by conjugation. More precisely, each $x \in P$ acts on $\bigoplus_L N(L)$ by $N([c_{x^{-1}}|_{{}^xL}])\colon N(L) \to N({}^xL)$, which we denote by $u \in N(L) \mapsto {}^xu \in N({}^xL)$. Define $I(N)(P)$ as the P-coinvariants, namely

$$I(N)(P) = \left(\bigoplus_{L \leq P, L \in \mathcal{X}} N(L) \right)_P.$$

Let $\pi_P\colon \bigoplus_L N(L) \to (\bigoplus_L N(L))_P$ denote the canonical projection. If $L \leq P \leq S$ are objects of $\mathcal{X}$, let $\pi_L^P = \pi_P \circ \iota_{N(L)}^{\oplus_L N(L)}\colon N(L) \to (\bigoplus_L N(L))_P$. Since we have taken P-coinvariants to define $I(N)(P)$, we have $\pi_{{}^xL}^P({}^xu) = \pi_L^P(u)$ for $u \in N(L)$ with $L \leq P$ in $\mathcal{X}$ and $x \in P$. If $Q \leq P \leq S$ are in $\mathcal{X}$, we define

$$\mathrm{t}_Q^P\colon I(N)(Q) \to I(N)(P)$$

by $\mathrm{t}_Q^P(\pi_L^Q(u)) = \pi_L^P(u)$ for $u \in N(L)$ with $L \leq Q$ in $\mathcal{X}$, and define

$$\mathrm{r}_Q^P\colon I(N)(P) \to I(N)(Q)$$

by

$$\mathrm{r}_Q^P(\pi_L^P(u)) = \sum_{x \in [Q \backslash P / L]_{\mathcal{X}}} \pi_{Q \cap {}^xL}^Q N([\iota_{Q \cap {}^xL}^{{}^xL}])({}^xu)$$

for $u \in N(L)$ with $L \leq P$ in $\mathcal{X}$. If $\varphi\colon P \to \varphi(P)$ is an isomorphism in $\mathcal{X}$, let

$$\mathrm{iso}(\varphi)\colon N(P) \to N(\varphi(P))$$

be given by $\mathrm{iso}(\varphi)(\pi_L^P(u)) = \pi_{\varphi(L)}^{\varphi(P)} N([\varphi|_L]^{-1})(u)$ for $u \in N(L)$ with $L \leq P$ in $\mathcal{X}$.

It is easy to show that $I(N)$ is a Mackey functor for $\mathcal{X}$ over k and that this construction is functorial. The only difference from the proof of [4, Lemma 2.10] is that our definition of r involves truncated double coset decomposition with respect to $\mathcal{X}$. To indicate how this is handled in our case, we prove the transitivity of r. Let $R \leq Q \leq P \leq S$ be in $\mathcal{X}$. Suppose $L \leq P$ is in $\mathcal{X}$ and $u \in N(L)$. We want to show that $\mathrm{r}_R^Q \mathrm{r}_Q^P(\pi_L^P(u)) = \mathrm{r}_R^P(\pi_L^P(u))$. The right hand side is

$$\sum_{x \in [R \backslash P / L]_{\mathcal{X}}} \pi_{R \cap {}^xL}^R N([\iota_{R \cap {}^xL}^{{}^xL}])({}^xu);$$

the left hand side is

$$\mathrm{r}_R^Q\left(\sum_{y\in[Q\backslash P/L]_{\mathcal{X}}}\pi_{Q\cap {}^yL}^Q N([\iota_{Q\cap {}^yL}^{{}^yL}])({}^yu)\right)$$
$$=\sum_{y\in[Q\backslash P/L]_{\mathcal{X}}}\sum_{z\in[R\backslash Q/Q\cap {}^yL]_{\mathcal{X}}}\pi_{R\cap {}^{zy}L}^R N([\iota_{R\cap {}^{zy}L}^{Q\cap {}^{zy}L}])({}^zN([\iota_{Q\cap {}^yL}^{{}^yL}])({}^yu))$$
$$=\sum_{y\in[Q\backslash P/L]_{\mathcal{X}}}\sum_{z\in[R\backslash Q/Q\cap {}^yL]_{\mathcal{X}}}\pi_{R\cap {}^{zy}L}^R N([\iota_{R\cap {}^{zy}L}^{{}^{zy}L}])({}^{zy}u).$$

Now we claim that the map

$$\{(y,z)\mid y\in[Q\backslash P/L], z\in[R\backslash Q/Q\cap {}^yL]\}\to[R\backslash P/L]$$

sending $(y,z)\mapsto zy$ is a bijection which restricts to a bijection

$$\{(y,z)\mid y\in[Q\backslash P/L]_{\mathcal{X}}, z\in[R\backslash Q/Q\cap {}^yL]_{\mathcal{X}}\}\to[R\backslash P/L]_{\mathcal{X}}.$$

To show this, it suffices to observe that, for $y\in[Q\backslash P/L]$, $z\in[R\backslash Q/Q\cap {}^yL]$, both $Q\cap {}^yL$ and $R\cap {}^{zy}L$ are objects of $\mathcal{X}$ if and only if $R\cap {}^{zy}L$ is in $\mathcal{X}$; this is because ${}^{z^{-1}}(R\cap {}^{zy}L)\subseteq Q\cap {}^yL$ as both R and z are contained in Q and χ is overgroup closed. This shows that r is transitive.

Finally we show that I is left adjoint to U^*. We define the unit of the adjunction

$$\eta\colon\mathrm{Id}_{\mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}},k\text{-}\mathsf{Mod})}\to U^*I$$

as follows. Let N be an object in $\mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod})$. For each $P\leq S$ in $\mathcal{X}$, we have a decomposition of k-modules

$$\left(\bigoplus_{L\leq P, L\in\mathcal{X}}N(L)\right)_P=N(P)\oplus\left(\bigoplus_{L<P, L\in\mathcal{X}}N(L)\right)_P.$$

Define

$$\eta_N(P)\colon N(P)\to U^*I(N)(P)$$

as the canonical injection of the above decomposition, i.e. $\eta_N(P)(u)=\pi_P^P(u)$ for $u\in N(P)$. Clearly each $\eta_N(P)$ commutes with r and iso, and η is natural in N. Thus η is a natural transformation. Similarly, we define the counit of the adjunction

$$\varepsilon\colon IU^*\to\mathrm{Id}_{\mathsf{Mack}_k(\mathcal{X})}$$

as follows. Let M be an object in $\mathsf{Mack}_k(\mathcal{X})$. For each $P\leq S$ in $\mathcal{X}$, define

$$\varepsilon_M(P)\colon IU^*(M)(P)\to M(P)$$

by $\varepsilon_M(P)(\pi_L^P(u))=\mathrm{t}_L^P(u)$ for $L\leq P$ in $\mathcal{X}$ and $u\in M(L)$. It is easy to check that each $\varepsilon_M(P)$ commutes with r, t and iso, and ε is natural in M. Thus ε is a natural transformation. Now clearly

$$\varepsilon I\circ I\eta=\mathrm{Id}_I,\qquad U^*\varepsilon\circ\eta U^*=\mathrm{Id}_{U^*}.$$

Thus η, ε are the unit and the counit of an adjunction. $\qquad\square$

Theorem 5.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let k be a field of characteristic p and let $N\in\mathsf{Fun}(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}, k\text{-}\mathsf{Mod})$. If*

$$n=\max\{m\mid\exists P_0<P_1<\cdots<P_m\leq S, P_i\in\mathcal{F}^c\,\forall i\},$$

then

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i N = 0 \qquad \forall i \geq n + 2.$$

Proof. From the proof of Theorem 5.1, the unit of the adjunction

$$\eta \colon \mathrm{Id}_{\mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod})} \to U^* I$$

restricts to an injective morphism $\eta_N \colon N \to U^* I(N)$ in $\mathsf{Fun}(\mathcal{O}(\mathcal{X})^{\mathrm{op}}, k\text{-}\mathsf{Mod})$ with cockernel $C(N)$ given by

$$C(N)(P) = \left(\bigoplus_{L < P, L \in \mathcal{X}} N(L) \right)_P$$

for each object P in $\mathcal{X}$. Since $U^* I(N)$ is acyclic by Theorem 4.1, from the short exact sequence

$$0 \to N \xrightarrow{\eta_N} U^* I(N) \to C(N) \to 0$$

we get an isomorphism

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i C(N) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}^{i+1} N$$

for all $i \geq 1$. Repeatedly applying the above isomorphism we get

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^{i+1} N \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)}^1 C^i(N)$$

By definition of $C(N)$, $C^i(N) = 0$ unless there is a chain of $\mathcal{F}$-centric subgroups $P_0 < P_1 < \cdots < P_i \leq S$. Thus $C^i(N) = 0$ for all $i \geq n+1$ and hence $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i N = 0$ for all $i \geq n + 2$. $\square$

6. Examples.

In this section we shall study acyclicity of functors over the centric orbit category for some families of fusion systems. We will use the procedure described in Proposition 4.5. So, for a fusion system $\mathcal{F}$ on a finite p-group S, we shall investigate restriction of simple Mackey functors to the centric orbit category, $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)}$, where $Q \leq S$ is not $\mathcal{F}$-centric and V is a simple $k\,\mathrm{Out}_{\mathcal{F}}(Q)$-module. We will often use Corollary 3.6 to deduce information on their values and Proposition 4.6 to check whether they are Mackey functors.

We will be mainly concerned with fusion systems over the extraspecial group p_+^{1+2} of order p^3 and exponent p for odd p and with the maximal nilpotency class 3-groups $B(3, r; 0, \gamma, 0)$ of 3-rank two. The original description of the latter family of groups is in [1, last paragraph p.88], where a group $B(3, r; \beta, \gamma, \delta)$ of order 3^r is defined for suitable values of three parameters β, γ and δ. We focus on the case where $\beta = \delta = 0$ as the exotic fusion systems of [11] occur only for these values of the parameters. The group $B = B(3, r; 0, \gamma, 0)$ is defined for $\gamma \in \{0, 1, 2\}$ if $r = 2k \geq 4$ and for $\gamma \in \{0, 1\}$ if $r = 2k + 1 \geq 5$. The group $B(3, r; 0, \gamma, 0)$ has 3-rank two except for $r = 4$ and $\gamma = 1$, when it is the wreath product of $\mathbb{Z}_3$ over itself and has 3-rank three. Throughout the paper, the notation $B(3, r; 0, \gamma, 0)$ implies that $r \geq 5$ or that $r = 4$ and $\gamma \in \{0, 2\}$.

The next lemma provides some information about the groups $B(3, r; 0, \gamma, 0)$.

Proposition 6.1. *Consider the group $B = B(3, r; 0, \gamma, 0)$ of order 3^r for $\gamma \in \{0, 1, 2\}$ if $r = 2k \geq 6$ and for $\gamma \in \{0, 1\}$ if $r = 2k + 1 \geq 5$. It is a semidirect product $B = \gamma_1 \rtimes \mathbb{Z}_3$ with the following properties.*

(a) *The subgroup γ_1 is characteristic abelian of rank two and contains the derived subgroup $B' = [B, B]$. The two generators s_1 and s_2 have orders 3^k and 3^k respectively if $r = 2k + 1$ and orders 3^k and 3^{k-1} respectively if $r = 2k$.*

(b) *Denote by s an order 3 generator of $\mathbb{Z}_3$. With respect to the ordered generators $\{s_1, s_2\}$ the action of s on γ_1 is described by the matrix $M^{r,\gamma}$ as follows*

$$M^{r,0} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, \quad M^{2k+1,1} = \begin{pmatrix} 1 & -3 \\ 1 & (-3)^{k-1} - 2 \end{pmatrix},$$

$$M^{2k,1} = \begin{pmatrix} 1 & -3((-3)^{k-2}+1) \\ 1 & -2 \end{pmatrix} \text{ and } M^{2k,2} = \begin{pmatrix} 1 & 3((-3)^{k-2}-1) \\ 1 & -2 \end{pmatrix}.$$

(c) *The center $Z(B)$ is of order 3 generated by $s_2^{3^{k-1}}$ if $r = 2k + 1$ and by $s_1^{3^{k-1}}$ if $r = 2k$.*

(d) *For each subgroup P of B one of the following holds.*
   (d1) *P is contained in γ_1. In this case, P is centric if and only if $P = \gamma_1$.*
   (d2) *P is not contained in γ_1. Set $K = P \cap \gamma_1$. Then K contains $Z(B)$ unless K is trivial. Moreover, $K = Z(B)$ if K has 3-rank one. In this case, P is centric if and only if K is nontrivial.*

Proof. Parts (a), (b) and (c) can be read from [11, A.2, A.9, A.11, A.12, A.13].

To prove part (d), let P be a subgroup of B. If P is not contained in γ_1 there is an element $sa \in P$ with $a \in \gamma_1$. This element acts on the normal subgroup $K = P \cap \gamma_1 \leq P$. As γ_1 is abelian, the action of a on K is trivial, and K must also be invariant under s, i.e., under the action of the matrix $M^{r,\gamma}$. Consider an element $\begin{pmatrix} c \\ d \end{pmatrix}$ of order 3 of K. It turns out that $M^{2k+1,\gamma}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \\ c+d \end{pmatrix}$ and that $M^{2k,\gamma}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c-3d \\ d \end{pmatrix}$ for all appropriate γ. Thus the center $Z(B)$ is in K if K is not trivial.

It remains to prove that if K has 3-rank one then it equals $Z(B)$. Note that if K has 3-rank one then it is cyclic. If $Z(B) < K$ then K must be generated by $s_2^{3^l}$ with $l \leq k - 2$ for odd r and by $s_1^{3^l}$ with $l \leq k - 2$ for even r by (c). Then it is easy to check using (b) that K cannot be s-invariant.

To deal with centricity, let $P \leq B$. If $P \leq \gamma_1$ then $\gamma_1 \leq C_B(P)$ and hence P cannot be centric unless it equals γ_1. If P is not contained in γ_1 then there is an element $sa \in P$ with $a \in \gamma_1$. Note that $C_B(P) \leq C_B(\langle sa \rangle)$. Using commutator identities and that $B' \leq \gamma_1$ it is easy to see that for any $b, c \in \gamma_1$ and $i \in \{1, 2\}$, $[s^i b, s^i c] = 1$ if and only if $c \in bZ(B)$. The result follows.   □

Remark 6.2. For the groups $B(3, 4; 0, 0, 0)$ and $B(3, 4; 0, 2, 0)$ the corresponding matrices are $M^{4,0} = \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$ and $M^{4,2} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and the conclusions (a), (c) and (d) in Proposition 6.1 hold as well.

As a corollary we obtain a classification of the $\mathcal{F}$-centric subgroups for any fusion system on $B(3, r; 0, \gamma, 0)$. This is finer than [11, Lemma 5.2], where just the subgroups which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical are described.

Corollary 6.3. *Let $\mathcal{F}$ be a a fusion system on $B = B(3, r; 0, \gamma, 0)$ and let P be a subgroup of B.*

(1) *If P is contained in γ_1 then P is $\mathcal{F}$-centric if and only if $P = \gamma_1$.*

(2) *Assume that P is not contained in γ_1 and set $K = P \cap \gamma_1$. Then P is $\mathcal{F}$-centric unless K is trivial or $K = Z(B)$ and $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$ is $\mathcal{F}$-conjugate to a subgroup of γ_1.*

Proof. (1) If P is $\mathcal{F}$-centric then it is also centric and hence by Proposition 6.1(d1) it equals γ_1. Now by Proposition 6.1(d) the only $\mathcal{F}$-conjugate of γ_1 is itself and we are done.

(2) In order for P to be centric, K must contain $Z(B)$ by Proposition 6.1(d2). If $|K| \geq 9$ then P is nonabelian and hence all $\mathcal{F}$-conjugates of P lie outside γ_1 and are centric by Proposition 6.1(d2). In this case P is $\mathcal{F}$-centric. If $|K| = 3$ then either $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $P \cong \mathbb{Z}_9$. Moreover, P might be $\mathcal{F}$-conjugate to P' not contained in γ_1 or to $P'' \leq \gamma_1$. The subgroup P' is centric by Proposition 6.1(d2) and $C_S(P'') = \gamma_1$. $\qquad\square$

Theorem 6.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $M = (M^*, M_*)\colon \mathcal{O}(\mathcal{F}) \to \mathbb{F}_p\text{-}\mathsf{mod}$ be a Mackey functor. Then*

$$\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)} M^*|_{\mathcal{O}(\mathcal{F}^c)} = 0 \qquad \forall i \geq 1$$

provided one of the following conditions hold.

(1) $p = 2$ and $S \cong \mathrm{D}_{2^n}$, SD_{2^n}, or Q_{2^n}.
(2) p odd and $S \cong p_+^{1+2}$ is the extraspecial group of order p^3 and exponent p.
(3) $p = 3$ and $S \cong B(3, r; 0, \gamma, 0)$ is a maximal nilpotency class 3-group of 3-rank two.
(4) S is resistant.

Proof. Consider a composition series of M. Let $S_{Q,V}$ be one of the composition factors where $Q \leq S$ is not $\mathcal{F}$-centric and V is a simple $k\,\mathrm{Out}_{\mathcal{F}}(Q)$-module. By Proposition 4.5, it suffices to show that $S_{Q,V}$ has vanishing higher limits. In cases (1) and (2) we will show that $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)} = 0$.

(1) The noncentric subgroups of S are precisely the cyclic subgroups except for the unique maximal cyclic subgroup. But the automorphism group of a cyclic 2-group is again a 2-group. Thus the only composition factor indexed by a non-$\mathcal{F}$-centric subgroup Q of S is of the form $S_{Q,k}$ where k is the trivial module. Now $S_{Q,k}(P) = 0$ for all $\mathcal{F}$-centric subgroup P of S by Corollary 3.6(2).

(2) The noncentric subgroups of S are precisely the cyclic subgroups of S, all of which have order p. Let P be a centric subgroup of S. Then P is an elementary abelian group of order p^2 or it is equal to $S = p_+^{1+2}$. Let L be any $\mathcal{F}$-conjugate of Q lying in P. For $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $C_P(L) = P$. For $P = S$, $C_S(L)$ is either $\mathbb{Z}_p \times \mathbb{Z}_p$ or S itself. In any case $S_{Q,V}(P) = 0$ by Corollary 3.6(1).

The proof of (3) requires a more meticulous analysis. By Corollary 6.3, Q is isomorphic to $\mathbb{Z}_{3^n}$ or $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^m}$ for some natural numbers n and m. Direct computation shows that for $Q \cong \mathbb{Z}_{3^n}$ or $Q \cong \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^m}$ with $n \neq m$ the hypothesis of Corollary 3.6(3) holds. Hence in these cases $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)} = 0$. In the remaining cases we show that $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)}$ is a Mackey functor using Proposition 4.6.

(3.1) **Case $Q \cong \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$ with $n \geq 2$:** By Proposition 6.1(d), $Q = \mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n} \leq \gamma_1$ is the only subgroup of S isomorphic to $\mathbb{Z}_{3^n} \times \mathbb{Z}_{3^n}$. Hence, by Proposition 3.2, for each $P \leq S$, either $Q \not\leq P$ and $S_{Q,V}(P) = 0$, or $Q \leq P$ and $S_{Q,V}(P) = \mathrm{tr}_Q^{N_P(Q)}(V)$. Assume that $S_{Q,V}(P) \neq 0$. Then by Corollary 3.6(1) we know that $C_P(Q) \leq Q$ and so $\gamma_1 \cap P = Q$. By Proposition 6.3 we conclude that if P is not $\mathcal{F}$-centric and $S_{Q,V}(P) \neq 0$ then $P = Q$ and that, if P is $\mathcal{F}$-centric and $S_{Q,V}(P) \neq 0$, then P is not contained in γ_1 and $P \cap \gamma_1 = Q$.

Now we prove that $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)}$ is a Mackey functor using Proposition 4.6. So let P and R be $\mathcal{F}$-centric subgroups such that $P \cap R$ is not $\mathcal{F}$-centric. By the

paragraph above we may assume that $P \cap R = Q$ and we have to check that the composition

$$\mathrm{tr}_Q^P(V) \xrightarrow{\mathrm{r}_Q^P} V \xrightarrow{\mathrm{t}_Q^R} \mathrm{tr}_Q^R(V)$$

is zero. Denote by sa and sb with $a, b \in \gamma_1$ elements in P and R not contained in Q. Note that the action of both elements on Q is identical as Q, a and b lie in the abelian group γ_1. This implies that in the earlier formula we can replace $\mathrm{tr}_Q^R(V)$ by $\mathrm{tr}_Q^P(V)$ and t_Q^R by t_Q^P to get

$$\mathrm{tr}_Q^P(V) \xrightarrow{\mathrm{r}_Q^P} V \xrightarrow{\mathrm{t}_Q^P} \mathrm{tr}_Q^P(V).$$

By Corollary 3.6(4) this composition is 0.

(3.2) **Case** $Q \cong \mathbb{Z}_3 \times \mathbb{Z}_3$: By Proposition 6.3, if Q is not contained in γ_1, it is $\mathcal{F}$-conjugate to a subgroup of γ_1 as Q is not $\mathcal{F}$-centric. So we may assume that $Q = \mathbb{Z}_3 \times \mathbb{Z}_3 \leq \gamma_1$, the unique elementary abelian subgroup of γ_1. Choose a subgroup P of S such that $S_{Q,V}(P) \neq 0$. By Proposition 3.2, $\mathrm{tr}_L^{N_P(L)}(^\alpha V) \neq 0$ for some $\mathcal{F}$-isomorphism $Q \xrightarrow{\alpha} L \leq P$. If $L \leq \gamma_1$ then $L = Q$, $C_P(Q) \leq Q$ by Corollary 3.6(1) and so $\gamma_1 \cap P = Q$. Assume now that L is not contained in γ_1. Then by Proposition 6.1(d2) there is a commutative diagram

$$
\begin{array}{ccccc}
K & \longrightarrow & P & \longrightarrow & \mathbb{Z}_3 \\
\uparrow & & \uparrow & & \| \\
Z(B) & \longrightarrow & L & \longrightarrow & \mathbb{Z}_3,
\end{array}
$$

where $K = P \cap \gamma_1$. If P is $\mathcal{F}$-centric, then $P > L$ and so $K > Z(B)$. By Proposition 6.1(d2), it follows that K contains Q. We conclude from Proposition 6.3 that if P is not $\mathcal{F}$-centric and $S_{Q,V}(P) \neq 0$ then P is $\mathcal{F}$-conjugate to Q and that if P is $\mathcal{F}$-centric and $S_{Q,V}(P) \neq 0$, then $Q \leq P$ and $P \not\leq \gamma_1$.

Next we prove that $S_{Q,V}|_{\mathcal{O}(\mathcal{F}^c)}$ is a Mackey functor using Proposition 4.6. Consider $\mathcal{F}$-centric subgroups P and R such that $P \cap R$ is not $\mathcal{F}$-centric. By the above paragraph we may assume that $P \cap R = Q$ and we have to check that the composition

$$S_{Q,V}(P) \xrightarrow{\mathrm{r}_Q^P} V \xrightarrow{\mathrm{t}_Q^R} S_{Q,V}(R)$$

is zero. Moreover, P, R contain subgroups P', R', respectively, such that $P' \cap \gamma_1 = R' \cap \gamma_1 = Q$ and $P' \not\leq \gamma_1$, $R' \not\leq \gamma_1$. By transitivity of r and t, we may further assume that $P = P'$, $R = R'$. By the description of the transfer t_Q^R in Proposition 3.2, we can factor the composition above as follows

$$S_{Q,V}(P) \xrightarrow{\mathrm{r}_Q^P} V \xrightarrow{\mathrm{tr}_Q^R} \mathrm{tr}_Q^R(V) \leq S_{Q,V}(R),$$

where $\mathrm{tr}_Q^R(V)$ is a direct summand of $S_{Q,V}(R)$ and $\mathrm{tr}_Q^R : V \to \mathrm{t}_Q^R(V)$ is now the trace map for the module V. As in the case (3.1), $\mathrm{tr}_Q^R(V) = \mathrm{tr}_Q^P(V)$ and $\mathrm{tr}_Q^R : V \to \mathrm{tr}_Q^R(V)$ is equal to $\mathrm{tr}_Q^P : V \to \mathrm{tr}_Q^P(V)$. Hence, we may assume that $P = R$ and we are done by Corollary 3.6(4).

The proof of (4) does not rely on Mackey functors. If S is resistant and $\mathcal{F}$ is a saturated fusion system over S then every morphism in $\mathcal{F}$ extends up to S. So for any functor $M : \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to k\text{-mod}$ the inverse limit $\varprojlim M$ is given by the invariants

$M(S)^{\mathrm{Out}_{\mathcal{F}}(S)}$. Hence the inverse limit functor $\varprojlim : \mathsf{Fun}(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}, k\text{-mod}) \to k\text{-mod}$ can be factored as follows

$$\mathsf{Fun}(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}, k\text{-mod}) \overset{\mathrm{ev}_S}{\to} k\,\mathrm{Out}_{\mathcal{F}}(S)\text{-mod} \overset{(-)^{\mathrm{Out}_{\mathcal{F}}(S)}}{\longrightarrow} k\text{-mod},$$

where the evaluation functor $F \mapsto \mathrm{ev}_S(F) = F(S)$ is an exact functor and the fixed points functor $W \mapsto W^{\mathrm{Out}_{\mathcal{F}}(S)}$ is left exact. The same argument as in [19, Exercise 2.3.7] gives an injective resolution of M in $\mathsf{Fun}(\mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}, k\text{-mod})$ consisting of functors whose values at S are free $k\,\mathrm{Out}_{\mathcal{F}}(S)$-modules. Moreover, free $k\,\mathrm{Out}_{\mathcal{F}}(S)$-modules are acyclic for the fixed point functor $(-)^{\mathrm{Out}_{\mathcal{F}}(S)}$ by Shapiro's lemma. Now a Grothendieck spectral sequence argument gives $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}^i M = H^i(\mathrm{Out}_{\mathcal{F}}(S); M(S))$. If $k = \mathbb{F}_p$ then $H^i(\mathrm{Out}_{\mathcal{F}}(S); M(S)) = 0$ for $i \geq 1$ as $\mathrm{Out}_{\mathcal{F}}(S)$ has order prime to p. $\qquad\square$

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